INVESTIGATION OF THE STABILITY OF A GYROSCOPE TAKING INTO ACCOUNT THE DRY FRICTION ON THE AXIS OF THE INNER CARDAN RING (GIMBAL)

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Lately, the exact theory of gyroscopic systems based on the application of the direct method of Liapunov, has been greatly developed; in particular, in the works of Chetaev [1], Rumiantsev [2] and Magnus [3].

With regard to the works of other authors who, in their investigations of the differential equations of motion of the gyroscope, often rejected certain terms without any proper justification. In the above mentioned papers the exact solutions of the differential equations are given, and the properties of the solutions are studied by means of construction of Liapunov's functions. Rumiantsev solved the question of viscous friction on the axes of suspension rings. In the paper under consideration an attempt is made to apply the direct method of Liapunov for the investigation of stability of certain motions of a gyroscope in a Cardan suspension, assuming dry friction on the axis of the gimbal.

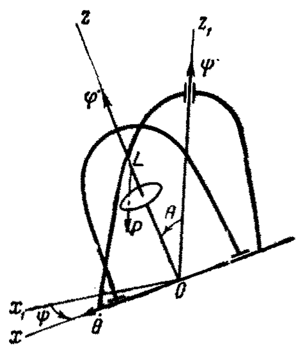

Fig. 1.

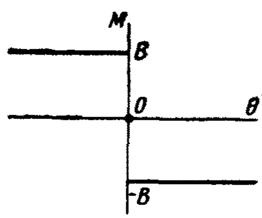

Fig. 2.

Consider a gyroscope in a Cardan suspension and introduce the following notations (Fig. 1). Let x_1, y_1, z_1 be a fixed coordinate system; x, y, z - a moving coordinate system which is rigidly connected with the gimbal (the x-axis is directed along the axis of the gimbal and the z-axis along the axis of the rotor), ψ - the angle of rotation of the outer ring, θ - the angle of rotation of the gimbal in the ring, ϕ - the angle of proper rotation [spin] of the gyroscope, L - the center of gravity of the system consisting of the gimbal and the rotor, P - the weight of the gyroscope and the gimbal; A, A, C - the moments of inertia of the gyroscope with respect to the axes; A_1, B_1, C_1 - the moments of inertia of the inner ring with respect to the same axes, I - the moment of inertia of the outer ring with respect to the z_1-axis. The axis z_1 of the outer ring is directed along the vertical. The distance OL is denoted by ζ. A moment of dry friction of the form $M = - B \operatorname{sign} \theta$, where $B > 0$, acts along the x-axis (Fig. 2). First assume that the friction arises only in the motion of the gyroscope, i.e. $M = 0$ for $\theta = 0$. In what follows we shall eliminate this restriction.

The general kinetic energy has the form

$$2T = \psi^{\cdot 2} (I + (A + B_1) \sin^2 \theta + C_1 \cos^2 \theta) + (A + A_1) \theta^{\cdot 2} + C (\varphi^\cdot + \psi^\cdot \cos \theta)^2$$

Since the variables ψ, θ, ϕ are independent holonomic coordinates, the equations of motion of the system can be written in the form of Lagrangian equations of the second kind

$$(A + A_1) \theta^{\cdot\cdot} - (A + B_1 - C_1) \psi^{\cdot 2} \sin \theta \cos \theta + C (\varphi^\cdot + \psi^\cdot \cos \theta) \psi^\cdot \sin \theta - P\zeta \sin \theta = M$$

$$\frac{d}{dt} [J\psi^\cdot + (A + B_1) \psi^\cdot \sin^2 \theta + C_1 \psi^\cdot \cos^2 \theta + C (\varphi^\cdot + \psi^\cdot \cos \theta) \cos \theta] = 0$$

$$\frac{d}{dt} (\varphi^\cdot + \psi^\cdot \cos \theta) = 0 \tag{1}$$

This system of equations admits two first integrals

$$\varphi^\cdot + \psi^\cdot \cos \theta = r_0 \tag{2}$$

$$J\psi^\cdot + (A + B_1) \psi^\cdot \sin^2 \theta + C_1 \psi^\cdot \cos^2 \theta + C r_0 \cos \theta = K \tag{3}$$

There is also a relation which is satisfied for all instants of time t, except $\theta(t) = 0$, i.e. for all t for which

$$\frac{d}{dt} M(t) = 0 \tag{4}$$

$$J\psi^{\cdot 2} + (A + A_1) \theta^{\cdot 2} + (A + B_1) \psi^{\cdot 2} \sin^2 \theta + C_1 \psi^{\cdot 2} \cos^2 \theta + 2P\zeta \cos \theta + C r_0^2 + 2M\theta = h_1$$

This relation is analogous to the law of conservation of energy which holds if there is no friction. We shall call it a "conditional" first integral.

Let us investigate the stability of the regular precession of the gyroscope:

$$\theta = \theta_0, \qquad \theta^\cdot = 0, \qquad \psi^\cdot = \Omega, \qquad r_0 = \omega$$

As it is seen from the first equation of system (1), this motion is possible only under the condition

$$[-(A+B_1-C_1)\,\Omega^2\cos\theta_0+C\Omega\omega-P\zeta]\sin\theta_0=M\,|_{\theta^{\cdot}=0}=0 \tag{5}$$

For the perturbed motion use the notations

$$\theta=\theta_0+\eta,\qquad \theta^{\cdot}=\eta^{\cdot}=\xi_1,\qquad \psi^{\cdot}=\Omega+\xi_2,\qquad r_0=\omega+\xi_3 \tag{6}$$

The equations of the perturbed motion, written to within terms of the second order, are

$$A_{11}\frac{d\xi_1}{dt}=M+B_{04}\xi_2+(-A_{44}+2\Omega B_{44})\,\eta+\Omega B_{34}\xi_3+2B_{44}\xi_2\eta+\frac{1}{2}B_{24}\xi_2^2+ \tag{7}$$

$$+B_{34}\xi_2\xi_3-\Omega B_{03}\xi_3\eta-\frac{1}{2}\Omega^2B_{24}\eta^2$$

$$\frac{d\xi_2}{dt}=-\frac{2B_{44}\eta+B_{24}\xi_2+B_{34}\xi_3+B_{04}}{B_{24}\eta+B_{02}}\,\xi_1,\qquad \frac{d\xi_3}{dt}=0,\qquad \frac{d\eta}{dt}=\xi_1$$

Here the following notations are used

$$A_{11}=A+A_1,\qquad A_{22}=(A+B_1)\sin^2\theta_0+C_1\cos^2\theta_0+J,\qquad A_{33}=C \tag{8}$$

$$A_{44}=(A+B_1-C_1)\,\Omega^2(\cos^2\theta_0-\sin^2\theta_0)-P\zeta\cos\theta_0,$$

$$A_{24}=4\,(A+B_1-C_1)\,\Omega\sin\theta_0\cos\theta_0,\qquad A_{02}=2\Omega\,[(A+B_1)\sin^2\theta_0+C_1\cos^2\theta_0+J]$$

$$A_{03}=2\omega C,\qquad A_{04}=2\,[(A+B_1-C_1)\,\Omega^2\cos\theta_0-P\zeta]\sin\theta_0-2M$$

$$B_{02}=(A+B_1)\sin^2\theta_0+C_1\cos^2\theta_0+J,\qquad B_{03}=C\cos\theta_0$$

$$B_{04}=[2\,(A+B_1-C_1)\,\Omega\cos\theta_0-C\omega]\sin\theta_0$$

$$B_{24}=2\,(A+B_1-C_1)\sin\theta_0\cos\theta_0,\qquad B_{34}=-C\sin\theta_0$$

$$B_{44}=(A+B_1-C_1)\,\Omega\,(\cos^2\theta_0-\sin^2\theta_0)-\frac{1}{2}C\omega\cos\theta_0$$

The above equations assume the following integrals (the first of which is a "conditional" integral):

$$V_1=A_{11}\xi_1^2+A_{22}\xi_2^2+A_{33}\xi_3^2+A_{44}\eta^2+A_{24}\xi_2\eta+A_{02}\xi_2+A_{03}\xi_3+A_{04}\eta$$

$$V_2=B_{44}\eta^2+B_{24}\xi_2\eta+B_{34}\xi_3\eta+B_{02}\xi_2+B_{03}\xi_3+B_{04}\eta,\qquad V_3=\xi_3 \tag{9}$$

An immediate application of the method of linear coupling of integrals for the construction of the Liapunov function, introduced by Chetaev [4], is not possible, since the knowledge of three integrals is not sufficient for the elimination of three linear terms in ξ_2, ξ_3, η.

Therefore, consider the auxiliary function

$$\Phi=2\,(MB_{04}\eta+MB_{02}\xi_2+MB_{24}\xi_2\eta)+N_1\xi_1^2+N_2\xi_2^2+N_3\xi_3^2+N_4\eta^2+2MB_{44}\eta^2+$$

$$+2MB_{34}\xi_3\eta \tag{10}$$

where N_1, N_2, N_3, N_4 are certain positive constants. Let us construct the following relations:

$$J_1=V_1-2\Omega V_2-2C\,(\omega-\Omega\cos\theta_0)\,V_3,\qquad J_2=B_{04}V_1-A_{04}V_2-(B_{04}A_{03}-B_{03}A_{04})\,V_3$$

i.e.

$$J_1 = A_{11}\xi_1^2 + A_{22}\xi_2^2 + A_{33}\xi_3^2 + (A_{44} - 2\Omega B_{44})\eta^2 - 2\Omega B_{34}\xi_3\eta - 2M\eta \tag{11}$$

$$J_2 = A_{11}B_{04}\xi_1^2 + A_{22}B_{04}\xi_2^2 + A_{33}B_{04}\xi_3^2 + (A_{44}B_{04} - A_{04}B_{44})\eta^2 + (A_{24}B_{04} - A_{04}B_{24})\xi_2\eta - A_{04}B_{34}\xi_3\eta + 2B_{02}M\xi_2$$

Each of these relations contains only one linear term. Now we can construct a relation connecting J_1, J_2 and Φ which does not contain linear terms

$$V = \Phi + B_{04}J_1 - J_2$$

The function V so obtained, which is a quadratic form in terms of the variables ξ_1, ξ_2, ξ_3, η, can be taken for a Liapunov function

$$V = N_1\xi_1^2 + N_2\xi_2^2 + N_3\xi_3^2 + N_4\eta^2 \tag{12}$$

First, this form is positive definite in the sense of Liapunov. Second, as it is easy to show, the total derivative of V with respect to t by virtue of equations (7) (for all t, except those which satisfy $\theta(t) = 9$) has the form

$$\frac{dV}{dt} = \frac{d\Phi}{dt} = 2\,\frac{N_1}{A_{11}}M\xi_1 + \xi_1(a_1\xi_1 + a_2\xi_2 + a_3\xi_3 + a_4\eta + \cdots) \tag{13}$$

where a_1, a_2, a_3, a_4 are certain constants and the dots stand for second order terms which have not been written out

Since the linear term $M\,\xi_1 = -\,B\,\xi_1 \operatorname{sign} \xi_1$ is a negative function, dV/dt is a function of fixed sign, opposite to that of V.

The proof of Liapunov's theorem on stability, presented suitably for our case, gives the following result: for an arbitrary number $A^0 > 0$ given in advance, there exists a number $\lambda > 0$ such that, if the initial perturbations in absolute value do not exceed λ for all $t \geqslant t_0$, except those for which $\theta(t) = 0$, then the perturbed trajectories do not deviate from the unperturbed trajectory by a distance ρ less than A^0, i.e. $\rho \leqslant A^0$.

The condition $\theta(t) = 0$ means that the perturbed trajectory is $\theta = \text{const}$; and since a trajectory in the sense of the problem represents a continuous curve, then the condition $\rho \leqslant A^0$ will be satisfied for all $t \geqslant t_0$. Hence the motion under consideration is stable in the sense of Liapunov.

Now consider the case when for $\theta = 0$ the moment of friction is different from zero, i.e. $M|_{\theta=0} = M_0 \neq 0$, where $|M_0| < B$. In such a case the regular precession of a gyroscope is possible only under the condition that the initial values are connected by the relation

$$[-(A + B_1 - C_1)\,\Omega^2\cos\theta_0 + C\Omega\omega - P\zeta]\sin\theta_0 = M_0 \tag{14}$$

The function $M(\theta)$ can be represented in the form of a sum of two terms (for example, if $M_0 > 0$):

$$M(\theta^{\cdot}) = M_1(\theta^{\cdot}) + M_0, \qquad M_1(\theta^{\cdot}) = \begin{cases} B - M_0 & \text{for } \theta^{\cdot} < 0 \\ 0 & \text{for } \theta^{\cdot} = 0 \\ -(B + M_0) & \text{for } \theta^{\cdot} > 0 \end{cases}$$

By virtue of the inequality $|M_0| < B$ the quantity $B - M_0$ is strongly positive. The equations for the perturbed motion are obtained in an analogous manner, and, as is easy to verify, have the same form as equations (7), the only difference being that the function M is replaced by M_1. The above given investigation remains in force also in this case. Thus, the condition $|M_0| < B$, i.e.

$$|[-(A + B_1 - C_1)\,\Omega^2 \cos\theta_0 + C\Omega\omega - P\zeta]\sin\theta_0| < B$$

is a sufficient condition for the stability of the regular precession of a gyroscope in the case of a dry friction moment of the form $M = -B$ sign θ, acting on the axis of the inner ring.

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